Relativistic three-particle dynamical equations: I. Theoretical development

Sadhan K. Adhikari
Instituto de Física Teórica, Universidade Estadual Paulista
01405 São Paulo, São Paulo, Brasil

T. Frederico
Instituto de Estudos Avançados, Centro Técnico Aeroespacial
12231 São José dos Campos, São Paulo, Brasil

Lauro Tomio
Instituto de Física Teórica, Universidade Estadual Paulista
01405 São Paulo, São Paulo, Brasil

Starting from the two-particle Bethe-Salpeter equation in the ladder approximation and integrating over the time component of momentum, we rederive three dimensional scattering integral equations satisfying constraints of relativistic unitarity and covariance, first derived by Weinberg and by Blankenbecler and Sugar. These two-particle equations are shown to be related by a transformation of variables. Hence we show how to perform and relate identical dynamical calculation using these two equations. Similarly, starting from the Bethe-Salpeter-Faddeev equation for the three-particle system and integrating over the time component of momentum, we derive several three dimensional three-particle scattering equations satisfying constraints of relativistic unitarity and covariance. We relate two of these three-particle equations by a transformation of variables as in the two-particle case. The three-particle equations we derive are very practical and suitable for performing relativistic scattering calculations.

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I. INTRODUCTION

The physical basis for going beyond non-relativistic potential scattering is secure for atomic processes where most phenomena are accurately described within quantum electrodynamics (QED). [1] In contrast to quantum chromodynamics (QCD) in the hadronic and nuclear cases, QED is a weakly coupled, non-confining field theory. It is relatively well understood how to calculate physical quantities using the QED. For example, two-charged particle scattering can be described by the covariant relativistic Bethe-Salpeter (BS) equation, [2] which has the following generic structure

$$T = V + VG_0T, \tag{1}$$

where the meaning and structure of the potential, V, and the Green function, G_0, are quite different than for simple potential scattering except the fact that V has only potential cuts, and G_0 contributes to the unitarity cuts of the t matrix, T. The initial, final, and intermediate states of Eq. (1) are defined in terms of four-momenta, and virtual processes are now depicted as occurring off the mass-shell rather than off the energy-shell.

It is still unclear on how to progress from QCD to practical collision integral equations for hadronic and nuclear processes. Therefore, most efforts to this end have employed a mixture of some type of meson-baryon field theory with phenomenology in order to obtain equations of the form (1), that presumably have a wider range of validity than the non-relativistic equations of the Lippmann-Schwinger (LS) type. [3] In this picture, if Eq. (1) suffices to define T uniquely, then V is necessarily an approximation. There has been a number of detailed calculations for hadron-hadron collisions in the so-called ladder approximation of Eq. (1), where V is approximated by the lowest order Feynman graphs. Also, many approximation schemes have originated from the BS equation (1). [4] The full BS equation for the few-hadron system correctly incorporates the fundamental constraints on the t matrix, such as unitarity, particle creation etc. The BS equation, however, does not treat crossing symmetry exactly. Moreover, in the ladder approximation the BS equation does not satisfy unitarity at all energies, as they always contain some multi-particle contributions but not all. In our treatment, however, we shall be limited to a consideration of a truncated Fock space with two or three particles and the ladder approximation to the BS equation satisfies unitarity below three- and four-particle thresholds.

The two-particle BS equation is a formal relation between vacuum expectation values of time ordered products of covariant fields. None of the covariant amplitudes in this equation is known, so one needs models for them. One can make various truncations that reduce the

BS equation to a three-dimensional form. This reduction could be done in any choice of variables.

The most practical set of equations derived from the BS equation for the two- and the three-hadron systems are the instant-form (constant-time) dynamical equations. [5–7] The use of instant-form variables leads to dynamical equations quite similar to the usual non-relativistic equations generated by Schrödinger dynamics both for the two- and the three-particle systems and has been used virtually in all numerical calculations of few-hadron scattering at medium energies. [1,5–10] The essence of the approaches leading to such equations is the replacement of the full four-dimensional Green function G_0 in the BS equation by an approximate three-dimensional one, consistent with conditions of two- and three-particle unitarity. This procedure reduces the full four-dimensional BS equations to three-dimensional LS-like equations [3] while preserving essential features like relativistic covariance, few-particle unitarity, and correct non-relativistic limit. Such dynamical equations were first suggested by Blankenbecler and Sugar [5] for the two- particle system and later studied and extended to the three-particle system by Aaron et al. [6] However, these approximate equations develop certain defects, not possessed by the original BS equation, such as, incorrect treatment of left-hand cuts. The potential in the resulting three-dimensional equation is provided phenomenologically in order to fit experimental results. Practically all recent relativistic few-hadron scattering calculations have employed these approximate three-dimensional dynamical scattering equations in the instant-form. [9,11,12]

The above three-dimensional reduction of the full four-dimensional BS equation could be implemented in other variables also, apart from the normal Schrödinger variables. Another set of variables is the light-front variables, which we define in Sec. II. Weinberg [13] was the first to write relativistically covariant dynamical integral equations for the two-particle system using light-front variables. The few-particle light-front dynamical equations are usually not derived from the BS equation as an approximation but obtained in an ad hoc fashion. [14,15] The resulting equations, like the instant-form equations, are three-dimensional with a phenomenological potential.

The angular momentum decomposition in light-front variables, even for potential scattering with spherically symmetric potential, is non-trivial and this fact has severely limited the applications of the light-front dynamics in actual physical problems of scattering.

In this paper we provide a general procedure for reducing four-dimensional relativistic scattering integral equations, satisfying unitarity, to three-dimensional forms. The present procedure consists of integrating over the time component of the momentum variable in the

intermediate state. In the case of the instant-form momentum variables this implies an integration over the variable, p_0, whereas in the case of the light-front variables the integration is to be performed over the equivalent variable, p_-. While performing this integration one assumes that the integrand - essentially, the potential and the t matrix - is independent of the time component of momentum. There are no other approximations involved. Consequently, as the original four-dimensional equations satisfy unitarity and relativity, so do the reduced ones.

Our main objective in this paper is to derive practical three-dimensional three-particle scattering equations. However, using the present procedure, as an excercise, first we derive dynamical equations for the two-particle system starting from the BS equation. For the two-particle system we recover the equations first derived by Weinberg [13] and by Blankenbecler and Sugar [5]. We show that these two types of equations are related by a transformation of variables and hence are equivalent to each other. This also guarantees that the two-particle Weinberg equation yield unitary and rotationally invariant result. This provides us with a recipe of performing dynamical two-hadron scattering calculations using the Weinberg equation for a specific angular momentum via the equivalent Blankenbecler-Sugar equation where the angular momentum decomposition is under control. In the instant-form one has the usual one-dimensional partial-wave scattering equation to solve, while in the light-front-form one has the three-dimensional equation to solve. We show that both approaches lead to the same result. Our finding holds good for any partial-wave potential, local or non-local. We have worked out the algebra for an S-wave separable potential. We emphasize that we have not resolved the fundamental difficulties with the angular momentum decomposition in light-front variables, but have found a way to bypass them in solving the Weinberg equation. With the present recipe the solution of the Weinberg and BlS equations are identical to each other.

Next we employ the present procedure to the study of the three-particle system employing two-particle separable potentials with an objective to derive new three-particle scattering equations satisfying constraints of relativistic unitarity and covariance. The separable form of two-particle interaction simulates a two-particle bound state or an isobar. In the presence of two-particle separable potentials the three-particle equations are very similar to the usual two-particle equations with non-local potentials. Hence the present derivation of the dynamical equations for the two-particle system from the BS equation is readily applicable to the three-particle system. We derive several instant-form and one light-front scattering equations for the three-particle system. Three-dimensional three-particle scattering equations

have also been proposed in Refs. [5,6,9,14–16] One of the present three-particle instant-form equations we derive is a consequence of using a three-particle relativistic propagator derived by Ahmadzadeh and Tjon [9] and is shown to be related to the present light-front equation by a transformation of variables. The present instant-form equations are distinct from the dynamical equations derived by Aaron, Amado, and Young [6].

The instant-form equations we derive are very suitable for performing numerical calculation as we demonstrate for the three-nucleon system in the following paper. [17]

The plan of the paper is as follows. In Sec. II we present the essential kinematics for the two-particle system, derive the instant-form and light-front equations for the two-particle system starting from the BS equation, and show how and under what conditions one can pass from one form to the other. This allows us to perform a numerical calculation using the light-front equation via the equivalent equation in the instant form. We illustrate these ideas using a separable two-particle potential. In Sec. III we generalize the treatment of Sec. II to the case of three-particles with separable two-particle potentials. In particular we derive new instant-form and light front equations for the three-particle system, each satisfying two- and three-particle unitarity. Finally, in Sec. IV we present some concluding remarks.

II. THE TWO-PARTICLE PROBLEM

Our principal interest in the present study is to develop practical three-dimensional scattering integral equations for the three-particle system, which can be used easily in numerical calculations. We present a numerical application using these equations in the following paper. [17] However, in this section we find it convenient to illustrate our approach for the two-particle system, which is well-understood, before taking up the challanging task of deriving three-dimensional three-particle equations in the following section.

A. Kinematics

The instant-form dynamics is treated in terms of the usual four vectors $(x_0, \vec{x})$ and $(p_0, \vec{p})$ in the configuration and momentum spaces, respectively. Here $\vec{x} \equiv (x_1, x_2, x_3)$, $\vec{p} \equiv (p_1, p_2, p_3)$, where x and p denote position and momentum components; x_0 is time and p_0 represents energy. The invariant length square of the four vector x is given by $x^2 = (x_0^2 - \vec{x}^2)$. We adopt units $\hbar = c = 1$. The light-front dynamics is treated in terms of equivalent momentum variables $(p_+, p_-, \vec{p}_\perp)$, with $\vec{p}_\perp \equiv (p_1, p_2)$, $p_+ = p_0 + p_3$, and $p_- = p_0 - p_3$. The on the mass shell condition for a particle of mass m in the two systems are given by

$$p_0^2 = \vec{p}^2 + m^2, \tag{2}$$

and

$$p_+ p_- = \vec{p}_\perp^2 + m^2, \tag{3}$$

in the instant-form and light-front formalisms, respectively. One also has the following useful identity

$$p^2 - m^2 = p_+(p_- - \frac{\vec{p}_\perp^2 + m^2}{p_+}). \tag{4}$$

For a system of two particles the invariant energy square, s, is given by

$$s = P_+ P_- - \vec{P}_\perp^2 \tag{5}$$

$$= \sum_i \frac{m_i^2 + \vec{p}_{i\perp}^2}{p_{i+}} P_+ - \vec{P}_\perp^2 \tag{6}$$

$$= \sum_i \frac{m_i^2 + \vec{p}_{i\perp}^2}{x_i} - \vec{P}_\perp^2, \tag{7}$$

where $x_i = p_{i+}/P_+$ is the momentum fraction of particle i, p_i and m_i denote the momentum and mass of individual particles, the index i labels the particles, $i = 1, 2$. The variable P refers to the total four momentum of the system of two particles, $P = p_1 + p_2$ and its components are defined in close analogy to those of the individual particle. It is convenient to define the relative momentum variable

$$\vec{k}_\perp = x_1 \vec{p}_{2\perp} - x_2 \vec{p}_{1\perp}, \tag{8}$$

where $x_2 = (1 - x_1)$. In terms of this variable the invariant energy square is given by

$$s = \frac{m_2^2}{x_2} + \frac{m_1^2}{x_1} + \frac{\vec{k}_\perp^2}{x_1 x_2}. \tag{9}$$

This equation has the same form in all frames and this is one of the advantages of working with the light-front variables.

In the case of the instant-form variables it is convenient to work in the c.m. system, so that $\vec{P} = 0$ and $P_0 = \sqrt{s}$, and define the relative four momentum p by $2p = p_1 - p_2$ in close analogy with non-relativistic kinematics. The individual particle momenta are hence given by $p_1 = P/2 + p$ and $p_2 = P/2 - p$. We shall always be working in the c.m. system in both instant-form and light-front formalisms.

For instant-form and light-front variables the invariant energy square in the c.m. frame is given by

$$s = [(\vec{p}^2 + m_1^2)^{1/2} + (\vec{p}^2 + m_2^2)^{1/2}]^2, \tag{10}$$

and

$$s = \frac{m_1^2 + \vec{p}_\perp^2}{x_1} + \frac{m_2^2 + \vec{p}_\perp^2}{x_2}. \tag{11}$$

With this discussion of kinematics for two noninteracting particles we shall now present a discussion of the dynamics of two interacting particles in the next subsection.

B. Instant-form equations

As we base the unified derivation on the BS equation it is convenient to write the approximate form of the BS equation for the two-hadron system in the form

$$t(q,k,s) = V(q,k,s) + \frac{i}{(2\pi)^4}\int d^4p \frac{V(q,p,s)t(p,k,s)}{(p_1^2 - m_1^2 + i0)(p_2^2 - m_2^2 + i0)}. \tag{12}$$

The usual derivation of the instant-form equations approximates the two-particle Green function

$$G_0(p,s) \equiv (p_1^2 - m_1^2 + i0)^{-1}(p_2^2 - m_2^2 + i0)^{-1} \tag{13}$$

by a three-dimensional one satisfying unitarity, sets all the particles (even in the intermediate state) on the mass shell, and assumes further that the potential and the t matrix are independent of the time component of momentum variables involved.

If we assume in Eq. (12) that the potential and the t matrix are independent of the time component of momenta variables, the dp_0 integration can be readily performed and one arrives at an instant-form equation which is identical to the three dimensional equation of Blankenbecler and Sugar and of Aaron, Amado, and Young. [5,6] No further approximations on the Green function is needed for the purpose. As the original equation (12) satisfy unitarity and relativity, so do the reduced ones.

The same procedure could be carried through in the light-front variables, too. This procedure allows us to establish an equivalence between the light front and instant-form dynamical equations not only for the two-particle system but also for the three-particle system.

For the two particles in the intermediate state in Eq. (12) we have

$$p_1^2 - m_1^2 = (p_0 + \frac{\sqrt{s}}{2})^2 - \omega_1^2, \tag{14}$$

and

$$p_2^2 - m_2^2 = (p_0 - \frac{\sqrt{s}}{2})^2 - \omega_2^2, \tag{15}$$

where $\omega_i^2 = \vec{p}^2 + m_i^2, i = 1, 2$, are the squares of energy of the two particles. Hence the last term in the BS equation (12) is written as

$$\begin{aligned} VG_0t = & \frac{i}{(2\pi)^4} \int d\vec{p} \int_{-\infty}^{\infty} dp_0 \frac{V(\vec{q}, \vec{p}, s) t(\vec{p}, \vec{k}, s)}{(p_0 + \sqrt{s}/2 + \omega_1 - i0)(p_0 + \sqrt{s}/2 - \omega_1 + i0)} \\ & \times \frac{1}{(p_0 - \sqrt{s}/2 + \omega_2 - i0)(p_0 - \sqrt{s}/2 - \omega_2 + i0)}, \end{aligned} \tag{16}$$

where we have assumed that the potential, V, and the t matrix are independent of the time component of the four momentum. The evaluation of the dp_0 integral in Eq. (16) is a straightforward exercise in the analysis of complex variables using the Cauchy theorem. If the contour of p_0 integration is taken along the real axis from $-\infty$ to ∞ and closed in the counterclockwise sense along a semicircle in the upper half of the complex p_0 plane at infinity this integral is readily evaluated to yield

$$\begin{aligned} VG_0t = & -\frac{1}{(2\pi)^3} \int d\vec{p} \Big[\frac{1}{(-2\omega_1)(-\sqrt{s} - \omega_1 + \omega_2)(-\sqrt{s} - \omega_1 - \omega_2 - i0)} \\ & + \frac{1}{(-2\omega_2)(\sqrt{s} + \omega_1 - \omega_2)(\sqrt{s} - \omega_1 - \omega_2 + i0)} \Big] V(\vec{q}, \vec{p}, s) t(\vec{p}, \vec{k}, s). \end{aligned} \tag{17}$$

This last result can easily be simplified and consequently the BS equation (12) reduces to

$$t(\vec{q}, \vec{k}, s) = V(\vec{q}, \vec{k}, s) + \frac{1}{(2\pi)^3} \int d\vec{p} V(\vec{q}, \vec{p}, s) \Big[\frac{\omega_1 + \omega_2}{2\omega_1\omega_2[s - (\omega_1 + \omega_2)^2 + i0]} \Big] t(\vec{p}, \vec{k}, s). \tag{18}$$

This equation was first derived explicitly by Aaron, Amado, and Young [6] for different mass particles using the procedure of Blankenbecler and Sugar. [5] Equation (18) is three dimensional, covariant and satisfies constraints of unitarity. After a partial-wave projection the solution of this equation is easily related to scattering phase shifts.

The energy denominator of Eq. (18) has two poles. One of them corresponds to the condition $\sqrt{s} = (\omega_1 + \omega_2)$, and represents the propagation of two particles in the intermediate state. The other corresponds to the condition $\sqrt{s} = -(\omega_1 + \omega_2)$, and represents the propagation of two antiparticles. The first pole is responsible for maintaining unitarity in the two particle sector. Usually, all three dimensional reductions of relativistic scattering equations involve propagation of antiparticle(s) in the intermediate state. However, the residue of the quantity in the square bracket in Eq. (18) at the pole corresponding to the particle propagation has to be the same in all formulations, which is the condition for maintaining unitarity

in the two-particle sector. Bakker et al. [15] have suggested to eliminate the propagation of antiparticles by evaluating the quantity in the square bracket of Eq. (18) at the pole corresponding to propagation of particles. Consequently, Eq. (18) reduces to

$$t(\vec{q},\vec{k},s) = V(\vec{q},\vec{k},s) + \frac{1}{(2\pi)^3}\int d\vec{p}V(\vec{q},\vec{p},s)\Big[\frac{1}{4\omega_1\omega_2[\sqrt{s}-(\omega_1+\omega_2)+i0]}\Big]t(\vec{p},\vec{k},s). \tag{19}$$

Equation (19) further simplifies to Eq. (2.36) of Ref. [15] for equal mass particles, when the quantity in the square bracket of this equation is evaluated at the pole corresponding to particle propagation.

C. Light-front equations

Now we shall carry on the reduction procedure of the last subsection using the light-front variables. For the two particles in the intermediate state in Eq. (12) we now have

$$p_1^2 - m_1^2 + i0 = (p_+ + \frac{\sqrt{s}}{2})\Big[(p_- + \frac{\sqrt{s}}{2}) - \frac{\vec{p}_\perp^2 + m_1^2 - i0}{p_+ + \frac{\sqrt{s}}{2}}\Big], \tag{20}$$

and

$$p_2^2 - m_2^2 + i0 = (-p_+ + \frac{\sqrt{s}}{2})\Big[(-p_- + \frac{\sqrt{s}}{2}) - \frac{\vec{p}_\perp^2 + m_2^2 - i0}{-p_+ + \frac{\sqrt{s}}{2}}\Big]. \tag{21}$$

If we use Eqs. (20) and (21), the BS equation (12) becomes

$$t(q,k,s) = V(q,k,s) + \frac{i}{2(2\pi)^4}\int d\vec{p}_\perp \int_{-\infty}^{\infty} dp_+ \int_{-\infty}^{\infty} dp_- \frac{V(q,p,s)t(p,k,s)}{(p_+ + \frac{\sqrt{s}}{2})[(p_- + \frac{\sqrt{s}}{2}) - \frac{\vec{p}_\perp^2 + m_1^2 - i0}{p_+ + \frac{\sqrt{s}}{2}}]}$$
$$\times \frac{1}{(-p_+ + \frac{\sqrt{s}}{2})[(-p_- + \frac{\sqrt{s}}{2}) - \frac{\vec{p}_\perp^2 + m_2^2 - i0}{-p_+ + \frac{\sqrt{s}}{2}}]}. \tag{22}$$

In Eq. (22) the factor of (1/2) before the integral is the Jacobian of the transformation of integration variables from $p \equiv (p_0, \vec{p})$ to $p \equiv (p_-, p_+, \vec{p}_\perp)$. Also, a typical momentum four vector in this equation, e.g., q, is considered to have components $(q_-, q_+, \vec{q}_\perp)$. These components are to be contrasted with the usual components $(q_0, \vec{q})$. Of the light-front momentum variables, q_- plays the role of the time component of the momentum four vector, and following the procedure of the last subsection we would like to perform the p_- integration in Eq. (22). In the context of the light-front variables this integration can be performed if we take the t matrix and the potential to be independent of the time component of momentum. We assume that this is the case. A similar assumption was made while performing the p_0 integration in Eq. (16).

We note that the dp_- integral in Eq. (22) can be transformed into a contour integral of the form

$$\oint dz \frac{1}{(z-z_1)(z-z_2)}, \tag{23}$$

where the contour goes along the real z axis from $x=-\infty$ to $x=\infty$ and is closed in the counterclockwise sense along a semicircle in the upper-half complex z plane at infinity, with x being the real part of z. The integral (23) contributes when the contour includes only one of the poles of the integrand. Assuming that it includes the pole $z=z_1$, the result of the integral (23) is $2\pi i/(z_1-z_2)$.

The condition of including one of the poles of p_- of Eq. (22) inside the contour sets for positive $\sqrt{s}$ the following limit on p_+: $\sqrt{s}/2 > p_+ > -\sqrt{s}/2$. The integral over p_- of Eq. (22) is now readily evaluated to yield

$$\begin{aligned} VG_0t = {} & \frac{1}{2}\frac{i}{(2\pi)^4}\int d\vec{p}_\perp \int_{-\sqrt{s}/2}^{\sqrt{s}/2} dp_+ \frac{-2\pi i V(q_+,\vec{q}_\perp;p_+,\vec{p}_\perp;s)t(p_+,\vec{p}_\perp;k_+,\vec{k}_\perp;s)}{(p_+ + \sqrt{s}/2)(-p_+ + \sqrt{s}/2)} \\ & \times \frac{1}{\sqrt{s} - \frac{\vec{p}_\perp^2 + m_2^2}{-p_+ + \frac{\sqrt{s}}{2}} - \frac{\vec{p}_\perp^2 + m_1^2}{p_+ + \frac{\sqrt{s}}{2}} + i0}. \end{aligned} \tag{24}$$

It is convenient to rewrite Eq. (24) in terms of the momentum fraction x_p defined by

$$x_p = \frac{p_+ + \sqrt{s}/2}{\sqrt{s}}, \tag{25}$$

with other momentum fractions x_q, x_k, etc. defined analogously. In terms of these momentum fractions Eq. (24) is easily simplified and the BS equation (12) is rewritten as

$$\begin{aligned} t(x_q,\vec{q}_\perp;x_k,\vec{k}_\perp;s) = {} & V(x_q,\vec{q}_\perp;x_k,\vec{k}_\perp;s) + \frac{1}{(2\pi)^3}\int_0^1 \frac{dx_p}{2x_p(1-x_p)}\int d\vec{p}_\perp \\ & \times \frac{V(x_q,\vec{q}_\perp;x_p,\vec{p}_\perp;s)t(x_p,\vec{p}_\perp;x_k,\vec{k}_\perp;s)}{s - \frac{\vec{p}_\perp^2+m_2^2}{1-x_p} - \frac{\vec{p}_\perp^2+m_1^2}{x_p} + i0}. \end{aligned} \tag{26}$$

Equation (26), first derived by Weinberg, [13] is the light-front counterpart of the relativistic LS-type equation (18) in the instant form. This equation is not rotationally invariant in light-front variables and hence the physical content and interpretation of this equation and its relation to Eq. (18) is unclear. Related to this is the treatment of angular momentum in Eq. (26). We shed light on some of these problems in the next subsection. Specifically, we show the equivalence between the physical contents of Eqs. (18) and (26). As an illustration we consider an S wave separable potential in order to clarify some of these issues.

D. Equivalence between light-front and instant-form equations

We begin this subsection by demonstrating the equivalence between the above two sets of equations (18) and (26). In order to achieve our goal we would like to find a possible transformation of variable, x_p to p_3, which relates these two equations without any other approximation, whatsoever. We used Eq. (25) to derive (26). In Eq. (26), however, x_p is just a variable of integration. We recall that we have two on the mass shell particles in the intermediate state of four momentum given by $(\omega_1, \vec{p})$ and $(\omega_2, -\vec{p})$ in the c. m. frame with $\omega_i = (\vec{p}^2 + m_i^2)^{1/2}$, i=1,2. Physically, x_p represents the momentum fraction of the first particle and should be given by

$$x_p = \frac{\omega_1 + p_3}{\omega_1 + \omega_2}. \tag{27}$$

We shall see that transformation (27) will take us from the light-front equation (26) to the instant form equation (18) with some plausible conditions on the potentials and the t matrices.

The Jacobian of the transformation (27) is given by

$$\frac{dx_p}{dp_3} = x_p(1 - x_p)\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}. \tag{28}$$

We also note that for x_p given by Eq. (27) one has the following identity

$$\frac{\vec{p}_\perp^2 + m_2^2}{1 - x_p} + \frac{\vec{p}_\perp^2 + m_1^2}{x_p} = (\omega_1 + \omega_2)^2. \tag{29}$$

If we use Eqs. (28) and (29) in the light-front equation (26), we arrive at the instant-form equation (18), provided that the potential and the t matrix transform according to:

$$V(x_q, \vec{q}_\perp; x_p, \vec{p}_\perp; s) \to V(\vec{q}, \vec{p}; s), \tag{30}$$

$$t(x_q, \vec{q}_\perp; x_p, \vec{p}_\perp; s) \to t(\vec{q}, \vec{p}; s). \tag{31}$$

This establishes the desired equivalence. The instant-form equation (18) is manifestly rotationally invariant. The present equivalence guarantees that the light-front equation (26), though not rotationally invariant in light-front variables, becomes so once transformed to instant form. In other words, the light-front equation (26) should yield rotationally invariant result in normal instant-form variables. We shall explicitly demonstrate this in the next subsection for a separable potential.

We emphasize that the physical contents of the two equations (18) and (26) are distinct. However, the potentials in these two equations are not derived from any consistent theory. They are phenomenological inputs, considered as ad hoc approximations to the potential of the BS equation, designed to generate experimental observables via the two types of equations. By construction, both yield relativistically covariant result with correct non-relativistic limit. This result is rotationally invariant and unitary, as we shall see below, when expressed in physical instant-form variables.

The light-front equation is not manifestly spherically symmetric, and this leads to complications in angular momentum projection and in the interpretation of solutions of this equation. The present equivalence leads to a recipe to bypass both these problems. One possibility is to first transform the light-front potential to the instant form via (30) and then solve the equivalent instant-form equation. The other possibility is to solve the three dimensional light-front equation without attempting a partial wave projection for each normal physical partial wave concerned. We shall see in the following that the two possibilities lead to the same result.

The main non-trivial question is how to effect the transformation (30). This is achieved through the use of Eq. (29), which gives the unique transformation ($x_p \to p_3$) that relates the potentials in the two frames. However, for different mass particles this transformation is quite involved. This is why in the following we specialize to the case of equal mass particles, $m = m_1 = m_2$, where Eq. (29) reduces to

$$\vec{p}^2 = \frac{\vec{p}_\perp^2 + m^2}{4x_p(1-x_p)} - m^2. \tag{32}$$

Equation (32) provides the unique transformation required to construct the light-front potential from the instant-form potential in each physical partial wave. The partial wave projection in Eq. (18) for the equal mass case can be readily carried through and one has the following partial wave form

$$t_l(q,k,s) = V_l(q,k,s) + \frac{1}{4(2\pi)^3}\int_0^\infty \frac{4\pi p^2 dp}{(p^2+m^2)^{1/2}} V_l(q,p,s)\frac{1}{(k^2-p^2+i0)} t_l(p,k,s), \tag{33}$$

where l denotes the partial wave. In the partial wave equations, such as Eq. (33) and in the following, the momentum labels p, q etc. denote the modulus of the corresponding three vectors. In arriving at Eq.(33) we have set the variable k on the energy shell too, so that $s = 4(m^2 + k^2)$. Though there are many ways of making the partial wave projection, we have written the partial wave equation in such a way that the full phase space, $4\pi p^2 dp$, appears in this equation. This has no relevance on our conclusions. However, it makes our case more

transparent, as Eq.(33) is the full equation for a potential with just a l wave component, and the full equation has been demonstrated to be equivalent to the light-front equation.

It is interesting to note that in the non-relativistic limit the factor $(p^2+m^2)^{1/2}$ in Eq. (33) reduces to m and this equation reduces to the usual partial wave LS equation.

For the sake of completeness, we observe that in the case of equal mass particles the light-front equation (26) reduces to

$$t(x_q,\vec{q}_\perp;x_k,\vec{k}_\perp;s) = V(x_q,\vec{q}_\perp;x_k,\vec{k}_\perp;s) + \frac{1}{2(2\pi)^3}\int_0^1 \frac{dx_p}{x_p(1-x_p)}\int d\vec{p}_\perp \\ \times \frac{V(x_q,\vec{q}_\perp;x_p,\vec{p}_\perp;s)t(x_p,\vec{p}_\perp;x_k,\vec{k}_\perp;s)}{s-\frac{\vec{p}_\perp^2+m_2}{x_p(1-x_p)}+i0}. \tag{34}$$

Given a spherically symmetric potential with component $V_l(q,p;s)$ in partial wave l, a numerical calculation with the partial wave form (33) can be performed in a routine fashion. The transformation (32) provides the unique prescription of constructing the light-front (LF) potential, to be employed in Eq. (34), from the instant-form (IF) potential for the physical partial wave l via

$$V_l^{(LF)}(x_q,\vec{q}_\perp;x_p,\vec{p}_\perp;s) = V_l^{(IF)}(q,p;s) \\ = V_l^{(IF)}\left(\left[\frac{\vec{q}_\perp^2+m^2}{4x_q(1-x_q)}-m^2\right]^{1/2},\left[\frac{\vec{p}_\perp^2+m^2}{4x_p(1-x_p)}-m^2\right]^{1/2};s\right), \tag{35}$$

where we have used an equation of the type (32) for each of the momentum variables. Also we have used the indices LF and IF on potentials in order to be more specific. The present demonstration of equivalence of the two sets of equations together with prescription (35) suggest two equivalent calculational schemes in the two approaches which would lead to identical results. The first possibility is to solve the one dimensional instant-form equation (33) with potential $V_l^{(IF)}(q,p;s)$, and the second possibility is to solve the three dimensional light-front equation (34) with the potential $V_l^{(LF)}(x_q,\vec{q}_\perp;x_p,\vec{p}_\perp;s)$ given explicitly by the right hand side of Eq. (35). Once the functional form of the instant-form potential $V_l^{(IF)}(q,p;s)$ is known the equivalent light-front potential could be readily found out via (35).

We note again that we have not resolved the difficulties with the angular momentum projection of the light-front equations. The light-front equation still continues to be three dimensional. We have provided a recipe which allows to perform scattering calculations employing partial-wave instant-form potential via BlS and Weinberg equations. Provided that the potentials in these two equations are related by Eq. (35), both the equations lead

to the same result. Our conclusions are quite general. But as they are quite subtle, a simple and specific example would help make our issue. And we do this in the next subsection considering the analytic separable S wave potential model.

E. S wave potential model: an illustration

Separable potentials have been successfully used in diverse situations. We consider the following S wave energy independent separable two-hadron potential in the momentum space to be used in the partial wave instant-form equation (33)

$$V_0(|\vec{M}_q|, |\vec{M}_p|, s) \equiv V(|\vec{M}_q|, |\vec{M}_p|) = \lambda g(|\vec{M}_q|^2) g(|\vec{M}_p|^2) \tag{36}$$

where $\vec{M}_p$ is a special vector and is a function of momenta p_1 and p_2 of the two particles. This is in fact the relative momenta of the two particles in the c. m. frame. In order to maintain Lorentz covariance of the equations, the best thing is to introduce a Lorentz invariant in place of $|\vec{M}_p|$. The Lorentz invariant, which reduces to the relative momentum square in the c. m. frame is $[(p_1 - p_2)/2]^2$. For equal mass particles in the c. m. frame $|\vec{M}_p|^2 = \vec{p}^2$. Then Eq. (36) can be rewritten in the c. m. frame as

$$V(|\vec{M}_q|, |\vec{M}_p|) = \lambda g(\vec{q}^2) g(\vec{p}^2). \tag{37}$$

We assume the following simple form factors for the potential

$$g(\vec{p}^2) = \frac{1}{(\vec{p}^2 + \beta^2)^{1/2}}. \tag{38}$$

With this potential the t matrix of equation (33) is readily given by

$$t_0(q, p, \vec{k}^2) = \tau(\vec{k}^2) g(\vec{q}^2) g(\vec{p}^2), \tag{39}$$

where

$$\tau^{-1}(\vec{k}^2) = \frac{1}{\lambda} - \frac{1}{8\pi^2} \int_0^\infty p'^2 dp' \frac{g^2(\vec{p'}^2)}{(\vec{p'}^2 + m^2)^{1/2}(\vec{k}^2 - \vec{p'}^2 - +i0)}. \tag{40}$$

In Eqs. (39) and (40) $\vec{k}$ is on the energy shell relative momentum in the c. m. frame: $s = 4(\vec{k}^2 + m^2)$.

Such a t matrix is very plausible when the two-particle interaction proceeds through coupling to a definite isobar in a particular partial wave only. To make the algebra simple we will assume that all the particles considered are spin and isospinless and the isobar occurs

in relative S wave only. If desired, these restrictions could be removed in principle without too much trouble. The function $\tau^{-1}(\vec{k}^2)$ should have the form $(s - m_I^2)$ where m_I is the dressed mass of the isobar. We note that the function $\tau^{-1}(\vec{k}^2)$ has a zero at the isobar mass, carries the scattering phase, and has the unitarity cut, whereas the function g should yield the left-hand cut.

With the form factor of Eq. (38) the integral in Eq. (40) could be evaluated to yield

$$\tau^{-1}(\vec{k}^2) = \frac{1}{\lambda} + \frac{1}{8\pi^2(\beta^2 + \vec{k}^2)}\Big[\frac{\beta}{(m^2 - \beta^2)^{1/2}} arctan \frac{(m^2 - \beta^2)^{1/2}}{\beta} + \frac{ik\pi}{2(m^2 + \vec{k}^2)^{1/2}} + \frac{k}{2(m^2 + \vec{k}^2)^{1/2}} ln \frac{(m^2 + \vec{k}^2)^{1/2} - k}{(m^2 + \vec{k}^2)^{1/2} + k}\Big]. \tag{41}$$

Equations (39) - (41) provide the usual solution to the problem via the partial wave instant-form equation (33). In Eq. (41) the symbol k denotes the modulus of the on shell relative three momentum $\vec{k}$ in the c. m. frame.

The same problem could also be solved and the same solution obtained via the full three dimensional light-front equation (26). The potential to be used in this equation, given by prescription (35), is written explicitly as:

$$V(x_q, \vec{q}_\perp; x_p, \vec{p}_\perp; s) = \lambda g(x_p, \vec{p}_\perp) g(x_q, \vec{q}_\perp), \tag{42}$$

where

$$g(x_p, \vec{p}_\perp) = \Big[\frac{\vec{p}_\perp^2 + m^2}{4x_p(1 - x_p)} - m^2 + \beta^2\Big]^{-1/2}. \tag{43}$$

This potential is separable and the solution of Eq (26) could be found by analytic means. The solution is given by

$$t(x_q, \vec{q}_\perp; x_p, \vec{p}_\perp; s) = \tau(\vec{k}^2) g(x_p, \vec{p}_\perp) g(x_q, \vec{q}_\perp), \tag{44}$$

where

$$\tau^{-1}(\vec{k}^2) = \frac{1}{\lambda} - \frac{1}{16\pi^3} \int_0^1 \frac{dx_p}{x_p(1 - x_p)} \int d\vec{p}_\perp \frac{g^2(x_p, \vec{p}_\perp)}{s - \frac{\vec{p}_\perp^2 + m^2}{x_p(1 - x_p)}}. \tag{45}$$

With the form factor given by Eq. (43) the integral in Eq. (45) could be analytically evaluated. We note that the integrand is independent of the angles of the vector $\vec{p}_\perp$. If a transformation of variables to $z = (\vec{p}_\perp^2 + m^2)/[x_p(1 - x_p)]$ is made and x_p is assumed to be constant, the integral over $p_\perp$ could be evaluated to yield

$$\tau^{-1}(\vec{k}^2) = \frac{1}{\lambda} + \frac{1}{16\pi^2}\frac{4}{(4\beta^2 - 4m^2 + s)}\int_0^1 dxln\frac{m^2 + 4(\beta^2 - m^2)x(1-x)}{m^2 - sx(1-x)}. \qquad (46)$$

Using integrals [2.733] and [2.736] of Ref. [18] the remaining integral could be evaluated and after some lengthy but straightforward algebra one arrives at the final rotationally invariant result (41). This also provides an explicit demonstration of rotational invariance of the light-front equation (34) in instant-form variables. Though the light-front equation is not rotationally invariant in light-front variables, it yields rotationally invariant result when expressed in instant-form variables.

With condition (32) the form factors (38) and (43) are identical. Hence, the two t matrices given by Eqs. (39) and (44) are also identical. This demonstrates the equivalence between the instant-form and the light-front equations for the present separable potential. Of course, the equivalence is valid for any spherically symmetric partial wave potential - local or non-local. In the instant form one has to solve the partial wave one dimensional integral equation (26), whereas in the light-front formalism one has to solve the three dimensional equation (34). Both procedures lead to the same result subject to condition (32).

III. THE THREE-PARTICLE PROBLEM

We now use the procedure of the last section to obtain a set of relativistic three-particle equations both in the instant-form and light-front formalisms. Again for simplicity we shall consider the case of three identical spin and isospinless bosons. In stead of considering the general problem of three particle scattering we shall consider the problem of elastic scattering of one particle from the bound state of the other two. We also assume an isobar in the two-particle system.

A. Instant-form equations

Just as in the two-particle case we assume a form for the four-dimensional relativistic Bethe-Salpeter-Faddeev three-particle equations for three identical spin and isospinless bosons, which we take as

$$T(q,k,s) = 2B(q,k,s) + \frac{2i}{(2\pi)^4}\int d^4p B(q,p,s)\tau(\sigma_p)T(p,k,s), \qquad (47)$$

with $\sigma_p \equiv (P-p)^2 = (\sqrt{s} - \omega_{\vec{p}})^2 - \vec{p}^2$, where P is the total four momentum of the system of three particles in the c. m. frame and is given by $(\sqrt{s}, 0, 0, 0)$. The factor 2 in the Born and

the homogeneous terms of Eq. (47) is due to symmetrization for identical bosons. Equation (47) is a straightforward generalization of the non-relativistic Amado model, which has been shown to be equivalent to the Faddeev equation for separable two-particle potential. We note that Eq. (47) is not derived from some fundamental theory, but its structure is postulated from those of the two-particle BS equation and the non-relativistic Amado model. [19] In Eq. (47) τ has the same meaning as in Eq. (39). The internal dynamics of the two-particle sub-system given by τ is supposed to be provided, so that Eq. (47) is an effective two-particle equation. Equation (47) is represented diagramatically in Fig. 1.

In the model three-particle equation (47) the Green function $\tau(\sigma_p)$ is designed to maintain two-particle unitarity. The three dimensional reduction of the Born term of Eq. (47) will be responsible for maintaining three-particle unitarity. There exists several unitary reductions of the Born term, [6,9], here we provide other distinct unitary reductions.

Next, with the three-dimensional Born term, as in the two-particle case, we would like to reduce Eq. (47) to a three dimensional form by performing the integration over the variable p_0. During the process of integration we assume that the t matrix, T, and the Born term are independent of this variable as in the last section. With this assumption the last term in Eq. (47) is written as

$$2B\tau T \equiv \frac{2i}{(2\pi)^4}\int d\vec{p}\int_{-\infty}^{\infty} dp_0 \frac{gg\tau(\sigma_p)T(\vec{p},\vec{k},s)}{(p^2-m^2+i0)[(P-q-p)^2-m^2+i0]}, \tag{48}$$

$$= \frac{2i}{(2\pi)^4}\int d\vec{p}\int_{-\infty}^{\infty} dp_0 \frac{gg\tau(\sigma_p)T(\vec{p},\vec{k},s)}{(p_0^2-\omega_{\vec{p}}^2+i0)[(\sqrt{s}-\omega_{\vec{q}}-p_0)^2-\omega_{\vec{p}+\vec{q}}^2+i0]}, \tag{49}$$

$$= \frac{2i}{(2\pi)^4}\int d\vec{p}\int_{-\infty}^{\infty} dp_0 \frac{gg\tau(\sigma_p)T(\vec{p},\vec{k},s)}{(p_0+\omega_{\vec{p}}-i0)(p_0-\omega_{\vec{p}}+i0)}$$

$$\times \frac{1}{(p_0+\omega_{\vec{q}}-\sqrt{s}+\omega_{\vec{p}+\vec{q}}-i0)(p_0+\omega_{\vec{q}}-\sqrt{s}-\omega_{\vec{p}+\vec{q}}+i0)}. \tag{50}$$

Equation (48) and the homogeneous term of Eq. (12) are quite similar. In both we have two single particle propagators. The difference is that in Eq. (12) this propagation occurs in the c. m. system of the two particles, whereas in Eq. (48) this propagation occurs in the c. m. system of the three particles. If we set $(P-q)=(\sqrt{s},0,0,0,)$ in Eq. (48), we arrive at the c. m. frame of two particles and Eqs. (48) and (12) become identical. Because of this, the general analysis and the final result in both cases are similar. However, the algebra is more involved in the three-particle case.

The dp_0 integral in Eq. (50) is quite similar to that in Eq. (16) and it is evaluated by closing the contour of p_0 integration in the counterclockwise sense along a semicircle in the upper half of the complex p_0 plane at infinity. The result is given by

$$2B\tau T = -\frac{2}{(2\pi)^3}\int d\vec{p} g g \tau(\sigma_p) T(\vec{p},\vec{k},s)$$
$$\times \Big[\frac{1}{(-2\omega_{\vec{p}})(\omega_{\vec{q}}-\omega_{\vec{p}}-\sqrt{s}+\omega_{\vec{p}+\vec{q}})(\omega_{\vec{q}}-\omega_{\vec{p}}-\sqrt{s}-\omega_{\vec{p}+\vec{q}})}$$
$$+\frac{1}{(-2\omega_{\vec{p}+\vec{q}})(\omega_{\vec{p}}-\omega_{\vec{q}}+\sqrt{s}+\omega_{\vec{p}+\vec{q}})(-\omega_{\vec{q}}-\omega_{\vec{p}}-\sqrt{s}-\omega_{\vec{p}+\vec{q}}+i0)}\Big], \tag{51}$$
$$= \frac{2}{(2\pi)^3}\int \frac{d\vec{p}}{2\omega_{\vec{p}}}\frac{g(\omega_{\vec{p}}+\omega_{\vec{p}+\vec{q}})g}{\omega_{\vec{p}+\vec{q}}[(\sqrt{s}-\omega_{\vec{q}})^2-(\omega_{\vec{p}}+\omega_{\vec{p}+\vec{q}})^2+i0]}\tau(\sigma_p)T(\vec{p},\vec{k},s). \tag{52}$$

In this equation only the relevant $i0$ which controls the three-particle unitarity has been shown. The three-dimensional reduction of Eq. (47) is usually written in the form [6]

$$T(\vec{q},\vec{k},s) = 2B(\vec{q},\vec{k},s) + \frac{2}{(2\pi)^3}\int \frac{d\vec{p}}{2\omega_p} B(\vec{q},\vec{p},s)\tau(\sigma_p)T(\vec{p},\vec{k},s). \tag{53}$$

Comparing Eq. (52) with the last term on the right-hand-side of Eq. (53) we identify the Born term of the three-dimensional scattering integral equation (53) as

$$B(\vec{q},\vec{p},s) = \frac{g(\omega_{\vec{p}}+\omega_{\vec{p}+\vec{q}})g}{\omega_{\vec{p}+\vec{q}}[(\sqrt{s}-\omega_{\vec{q}})^2-(\omega_{\vec{p}}+\omega_{\vec{p}+\vec{q}})^2+i0]}. \tag{54}$$

Equations (53) and (54) are the desired three-dimensional instant-form scattering integral equations. These equations follow from the use of a relativistic three-particle propagator suggested by Ahmadzadeh and Tjon. [9] However, Eqs. (53) and (54) have never appeared explicitly in this form before.

Aaron, Amado, and Young [6] derived Eq. (53) with the Born term

$$B(\vec{q},\vec{p},s) = \frac{g(\omega_{\vec{p}+\vec{q}}+\omega_{\vec{p}}+\omega_{\vec{q}})g}{\omega_{\vec{p}+\vec{q}}[s-(\omega_{\vec{p}+\vec{q}}+\omega_{\vec{p}}+\omega_{\vec{q}})^2+i0]}. \tag{55}$$

Both sets - Eqs. (53) and (54), and Eqs. (53) and (55) - constitute linear, three-dimensional, rotationally invariant and Lorentz-covariant integral equations for the elastic scattering of one particle from the bound state (isobar) of the other two. They satisfy, by construction, two- and three-particle unitarity. After a partial-wave projection their solution can be easily related to scattering phase-shifts in usual fashion.

The factors g's in Eqs. (54) and (55) and in the following are the two-particle potential form factors as in Eq. (36). The arguments of these form factors have been suppressed to save space. We have noted in Sec. II E that the argument should be invariant and be the square of the relative momentum in the c. m. frame. In the instant form these arguments are explicitly given by [12]

$$\mathcal{P}^2 = (\omega_{\vec{q}}+\omega_{\vec{p}+\vec{q}})^2/4 - p^2/4 - m^2, \tag{56}$$

$$\mathcal{Q}^2 = (\omega_{\vec{p}} + \omega_{\vec{p}+\vec{q}})^2/4 - q^2/4 - m^2. \tag{57}$$

The denominator in Eq. (55) has two poles for $\sqrt{s} = \pm(\omega_{\vec{p}+\vec{q}} + \omega_{\vec{p}} + \omega_{\vec{q}})$. The plus sign refers to propagation of three particles and is alone responsible for maintaining the desired unitarity in the three-particle sector. The minus sign refers to propagation of three antiparticles. Note that the pole corresponding to the propagation of antiparticles does not materialize in the physical region: $s > 0$.

The present Born term (54) has two poles, too. One of them given by $\sqrt{s} = (\omega_{\vec{p}+\vec{q}}+\omega_{\vec{p}}+\omega_{\vec{q}})$ refers to the propagation of three particles and should be responsible for maintaining three-particle unitarity. The other pole is given by $\sqrt{s} = (\omega_{\vec{q}} - \omega_{\vec{p}+\vec{q}} - \omega_{\vec{p}})$ and represents the propagation of a particle and two antiparticles. Note that the pole corresponding to the propagation of antiparticles does not materialize in the scattering region: $s > 3m$. The requirement of the constraint of relativistic unitarity for these Born terms is to possess the same residue at the pole for three-particle propagation at $\sqrt{s} = (\omega_{\vec{p}+\vec{q}} + \omega_{\vec{p}} + \omega_{\vec{q}})$. This residue of the two Born terms (54) and (55) at the pole corresponding to the propagation of three particles are easily shown to be the same ($= gg/2\omega_{\vec{q}+\vec{p}}$).

There is another three-dimensional reduction which has all the desired features and does not allow antiparticle propagation in the physical scattering region. This reduction is given by

$$B(\vec{q},\vec{p};s) = \frac{g(\omega_{\vec{q}} + \omega_{\vec{p}})g}{\omega_{\vec{q}+\vec{p}}[(\sqrt{s} - \omega_{\vec{q}+\vec{p}})^2 - (\omega_{\vec{q}} + \omega_{\vec{p}})^2 + i0]}. \tag{58}$$

This Born term has two poles, too. One of them given by $\sqrt{s} = (\omega_{\vec{p}+\vec{q}}+\omega_{\vec{p}}+\omega_{\vec{q}})$ refers to the propagation of three particles and is be responsible for maintaining three-particle unitarity. The other pole is given by $\sqrt{s} = (\omega_{\vec{p}+\vec{q}} - \omega_{\vec{p}} - \omega_{\vec{q}})$ and represents the propagation of two antiparticles and a particle. The residue of the two Born terms (55) and (58) at the pole corresponding to the propagation of three particles are easily shown to be the same. Again, the propagation of the antiparticle does not materialize in the physical scattering region in the case of the Born term (58).

In the three-particle Born term we could suppress the antiparticle propagation in the intermediate state. This is achieved by evaluating the denominator corresponding to antiparticle propagation at the pole corresponding to particle propagation. Mathematically, this will constitute in exhibiting the pole for three-particle propagation and the residue. Consequently, the Born terms (54), (55) and (58) reduce to the following minimal form

$$B(\vec{q},\vec{p},s) = \frac{gg}{2\omega_{\vec{p}+\vec{q}}[\sqrt{s} - \omega_{\vec{p}} - \omega_{\vec{q}} - \omega_{\vec{p}+\vec{q}} + i0]}. \tag{59}$$

This is the simplest Born term which when used in Eq. (53) will yield unitary result and which does not allow propagation of antiparticles in the intermediate state.

The Born terms (58) and (59) are worth investigating analytically and numerically.

Next, let us consider the non-relativistic limit of these Born terms. In this limit the denominator, $(\sqrt{s}-\omega_{\vec{p}+\vec{q}}-\omega_{\vec{p}}-\omega_{\vec{q}})$, corresponding to three-particle propagation reduces to $[E-\{(\vec{p}+\vec{q})^2+\vec{p}^2+\vec{q}^2\}/2m]$, where E is the kinetic energy of the three-particle system in the c. m. frame. In the remaining factors of the Born term, ω reduces to mass m and $\sqrt{s}$ reduces to $3m$. The arguments of the form factors g's, given by Eqs. (56) and (57), reduce to $(\vec{p}/2+\vec{q})^2$ and $(\vec{q}/2+\vec{p})^2$, respectively. Consequently, all these Born terms reduce to the usual non-relativistic Amado model [19] Born term given by

$$B(\vec{q},\vec{p},E) \sim \frac{g[(\vec{p}/2+\vec{q})^2]g[(\vec{q}/2+\vec{p})^2]}{2mE-(\vec{p}+\vec{q})^2-\vec{p}^2-\vec{q}^2+i0}. \tag{60}$$

The phase space of Eq.(53) reduces to the usual non-relativistic phase space $d\vec{p}/[2m(2\pi)^3]$ in this limit, and the present set of equations reduces to the usual Amado model. [19]

Aaron, Amado, and Young [6] have shown how to calculate breakup amplitudes from a solution of similar equations. They also have demonstrated how to include spin and fermions in the formalism. Similar equations have frequently been used in pion-nucleon and pion-deuteron scattering problems. [11] Following their procedure we could extend these equations to more realistic situations.

B. Light-front equations

Next we would like to perform the reduction procedure of the last subsection in the light-front variables. Specifically, as in Sec. IIC we would like to perform the p_- integration in Eq. (47) and write a three dimensional integral equation in terms of variables $(p_+,\vec{p}_\perp)$ or $(x_p,\vec{p}_\perp)$. As usual, during the integration we assume that the t matrix, T, is independent of the variable p_-.

For the two particles in the intermediate state in the last term in Fig. 1, we have

$$p^2-m^2+i0 = p_+\left(p_- - \frac{\vec{p}_\perp^2+m^2-i0}{p_+}\right) \tag{61}$$

and

$$(P-q-p)^2-m^2+i0 = (\sqrt{s}-q_+-p_+)\left[\sqrt{s}-q_--p_- - \frac{(\vec{p}_\perp+\vec{q}_\perp)^2+m^2-i0}{\sqrt{s}-q_+-p_+}\right]. \tag{62}$$

If we use equations (61) and (62), the last term in Eq.(47) is written explicitly as

$$2B\tau T \equiv \frac{2i}{2(2\pi)^4}\int d\vec{p}_\perp \int_{-\infty}^{\infty} dp_+ \int_{-\infty}^{\infty} dp_- \frac{g\tau(\sigma_p)gT(p_+,\vec{p}_\perp;k_+,\vec{k}_\perp;s)}{p_+(p_- - \frac{\vec{p}_\perp^2+m^2-i0}{p_+})}$$
$$\times \frac{1}{(\sqrt{s}-q_+-p_+)\left[\sqrt{s}-q_- - p_- - \frac{(\vec{p}_\perp+\vec{q}_\perp)^2+m^2-i0}{\sqrt{s}-q_+-p_+}\right]}. \tag{63}$$

In Eq. (63), as in Eq. (22), the factor of (1/2) before the integral is the Jacobian of the transformation of integral variables.

The integral in Eq. (63) has the same structure as that in Eq. (22), and the integral over p_- is performed by the technique of contour integration in a similar fashion in the complex p_- plane, considering a contour along the real axis from $-\infty$ to ∞ and closing it in the counterclockwise sense along a semicircle at infinity. After performing this contour integration Eq. (63) reduces to

$$2B\tau T = \frac{2i}{(2\pi)^4}\int d\vec{p}_\perp \int_0^{\sqrt{s}-q_+} \frac{dp_+}{2p_+}\frac{-2\pi i g\tau(\sigma_p)gT(p_+,\vec{p}_\perp;k_+,\vec{k}_\perp;s)}{(\sqrt{s}-q_+-p_+)}$$
$$\times \frac{1}{\left[\sqrt{s}-q_- - \frac{\vec{p}_\perp^2+m^2}{p_+} - \frac{(\vec{p}_\perp+\vec{q}_\perp)^2+m^2}{\sqrt{s}-q_+-p_+} + i0\right]}. \tag{64}$$

The limits on the integral over p_+ in Eq.(64) appears, as in Eq. (24), as a consequence of including a single pole in the contour. (See, discussion related to Eq. (23) in Sec. IIC.)

If we set the external particle with momentum q on the mass shell, as in Eq. (3), Eq. (64) can be rewritten as

$$2B\tau T = \frac{2}{(2\pi)^3}\int d\vec{p}_\perp \int_0^{\sqrt{s}-q_+} \frac{dp_+}{2p_+}\left[\frac{gg}{(\sqrt{s}-q_+-p_+)[\sqrt{s}-\frac{\vec{q}_\perp^2+m^2}{q_+}-\frac{\vec{p}_\perp^2+m^2}{p_+}-\frac{(\vec{p}_\perp+\vec{q}_\perp)^2+m^2}{\sqrt{s}-q_+-p_+}+i0]}\right]$$
$$\times \tau(\sigma_p)T(p_+,\vec{p}_\perp;k_+,\vec{k}_\perp;s). \tag{65}$$

Equation (65) can also be written in terms of the momentum fractions, as in the two-particle case, conveniently defined by

$$x_p = p_+/\sqrt{s}, \tag{66}$$

and

$$x_q = q_+/\sqrt{s}. \tag{67}$$

Using these momentum fractions Eq. (65) can be rewritten as

$$2B\tau T = \frac{2}{(2\pi)^3}\int_0^{1-x_q} \frac{dx_p}{2x_p}\int d\vec{p}_\perp\left[\frac{gg}{(1-x_p-x_q)[s-\frac{\vec{q}_\perp^2+m^2}{x_q}-\frac{\vec{p}_\perp^2+m^2}{x_p}-\frac{(\vec{p}_\perp+\vec{q}_\perp)^2+m^2}{1-x_q-x_p}+i0]}\right]$$
$$\times \tau(\sigma_p)T(x_p,\vec{p}_\perp;x_k,\vec{k}_\perp;s). \tag{68}$$

It is tempting to define the quantity in the square bracket of Eq. (68) as the Born term of the following scattering integral equation

$$T = B + B\tau T, \tag{69}$$

as in the last subsection. But this identification is not quite to the point as Eq. (69) with the homogeneous term (68) is of the Volterra type and not of the usual Fredholm type. A standard Fredholm equation could be derived by a transformation of variables.

It is convenient to rewrite Eq. (68) in terms of new variable, x'_p, defined by

$$x'_p = \frac{x_p}{1 - x_q}. \tag{70}$$

In terms of this new variable Eq. (68) is rewritten as

$$2B\tau T = \frac{2}{(2\pi)^3} \int_0^1 \frac{dx'_p}{2x'_p(1-x'_p)} \int d\vec{p}_\perp \left[\frac{gg}{(1-x_q)(s - \frac{\vec{q}_\perp^2 + m^2}{x_q}) - \frac{\vec{p}_\perp^2 + m^2}{x'_p} - \frac{(\vec{p}_\perp + \vec{q}_\perp)^2 + m^2}{1 - x'_p} + i0} \right]$$
$$\times \tau(\sigma_p) T(x_p, \vec{p}_\perp; x_k, \vec{k}_\perp; s). \tag{71}$$

The quantity τ of Eq. (71) plays the role of the two-particle Green function. However, Eq. (71) by itself can not define the Born term of the scattering integral equation. The three-particle scattering equation should have the following generic form

$$T(x_q, \vec{q}_\perp; x_k, \vec{k}_\perp; s) = B(x_q, \vec{q}_\perp; x_k, \vec{k}_\perp; s) + \frac{2}{(2\pi)^3} \left[\int_0^1 \frac{dx'_p}{2x'_p(1-x'_p)} \int d\vec{p}_\perp \frac{1}{f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s)} \right]$$
$$\times B(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s) \tau(\sigma_p) T(x'_p, \vec{p}_\perp; x_k, \vec{k}_\perp; s), \tag{72}$$

with the Born term given by

$$B(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s) = \frac{g f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s) g}{(1-x_q)(s - \frac{\vec{q}_\perp^2 + m^2}{x_q}) + \frac{\vec{p}_\perp^2 + m^2}{x'_p} + \frac{(\vec{p}_\perp + \vec{q}_\perp)^2 + m^2}{1 - x'_p} + i0}, \tag{73}$$

where $f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s)$ is a function yet to be determined. This function is cancelled in the homogeneous term of Eq. (72). This function and hence the Born term can be uniquely determined by claiming the equivalence between the instant-form three-particle equations (53) and (54), and light-front three-particle equations (72) and (73). Equations (72) and (73) are the three-particle light-front equations which we seek. The homogeneous version of this equation has recently been solved numerically in a simple model. [16] We also note that Eqs. (72) and (73) are not rotationally invariant in light-front variables.

We shall show by a transformation of variable that the Born term and the dynamics represented by Eqs. (72) and (73) are essentially the same as those of the instant-form equations (53) and (54).

C. Equivalence between light-front and instant-form equations

In the two-particle problem starting from the BS equation (12) we derived the instant-form and the light-front equations, (18) and (26) by integrating over the variables p_0 and p_-, respectively. It was demonstrated that these two forms of equations were equivalent. We now carry on the same procedure in the case of the three-particle system. By claiming the equivalence between the light-front and instant-form equations we determine the Born term (73).

In the present case we are really considering the dynamics of two on the mass shell particles in the three-particle c. m. system. The momentum of these two particles are given by $(\omega_{\vec{p}}, \vec{p})$ and $(\omega_{\vec{p}+\vec{q}}, -\vec{p}-\vec{q})$. The momentum fraction of the first particle x'_p is given by

$$x'_p = \frac{\omega_{\vec{p}} + p_3}{\omega_{\vec{p}} + \omega_{\vec{p}+\vec{q}} - q_3}. \tag{74}$$

We shall see that this transformation, as in the two-particle case, will relate the three-particle instant-form and light-front equations. The Jacobian of this transformation is given by

$$\frac{dx'_p}{dp_3} = x'_p(1-x'_p)\frac{\omega_{\vec{p}} + \omega_{\vec{p}+\vec{q}}}{\omega_{\vec{p}}\omega_{\vec{p}+\vec{q}}}. \tag{75}$$

Equations (74) and (75) should be compared with corresponding equations (27) and (28) in the two-particle case. They are quite similar as both represent essentially two-particle dynamics. Recalling that x_q is defined by Eq. (67), the Born term (73) is rewritten as

$$B(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s) = \frac{g f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s) g}{(\sqrt{s} - \omega_{\vec{q}} - q_3)(\sqrt{s} - \omega_{\vec{q}} + q_3) - \frac{\vec{p}_\perp^2 + m^2}{x'_p} - \frac{(\vec{p}_\perp + \vec{q}_\perp)^2 + m^2}{1 - x'_p} + i0}. \tag{76}$$

If we use transformation (74) it is straightforward to show that

$$\frac{\vec{p}_\perp^2 + m^2}{x'_p} + \frac{(\vec{p}_\perp + \vec{q}_\perp)^2 + m^2}{1 - x'_p} = (\omega_{\vec{p}} + \omega_{\vec{p}+\vec{q}})^2 - q_3^2. \tag{77}$$

Provided that under the transformation (74) the function $f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s)$ transforms according to

$$f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s) \to \frac{\omega_{\vec{p}} + \omega_{\vec{p}+\vec{q}}}{\omega_{\vec{p}+\vec{q}}}, \tag{78}$$

with Eq. (77) the Born term of Eq. (76) reduces to

$$B(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s) \to \frac{g(\omega_{\vec{p}} + \omega_{\vec{p}+\vec{q}})g}{\omega_{\vec{p}+\vec{q}}[(\sqrt{s} - \omega_{\vec{q}})^2 - (\omega_{\vec{p}} + \omega_{\vec{p}+\vec{q}})^2 + i0]}, \tag{79}$$

which is identical to the Born term (54). We note that there is not only a one-to-one correspondence between the Born terms of Eqs. (54) and (73) under transformation (74), but because of Eqs. (75) and (78) the phase spaces of these two equations are also identical, explicitly

$$\frac{dx'_p}{2x'_p(1-x'_p)} d\vec{p}_\perp \frac{1}{f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s)} \to \frac{d\vec{p}}{2\omega_{\vec{p}}}. \tag{80}$$

This proves the complete equivalence between the three-particle light-front and instant-form dynamical equations, provided that the function $f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s)$ obeys Eq. (78) under this transformation. Hence one can perform a numerical solution of the light-front dynamical equations (72) and (73) via the equivalent instant-form equations (54) and (53). The instant-form equations (53) and (54) are rotationally invariant. The present demonstration of equivalence guarantees that, as in the two-particle case, the light-front equations (72) and (73), though not manifestly rotationally invariant, will yield unitary and rotationally invariant results in usual instant-form variables. In short, we have derived the sets of equations given equivalently, by Eqs. (53) and (54), and by Eqs. (72) and (73), in the instant-form and light-front formalisms, respectively.

However, we could not find an explicit expression for the function $f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s)$. We have Eq. (78) subject to the transformation (74).

Fuda [14] also derived three-particle light-front equations for spinless meson-baryon scattering using two-particle isobars. The homogeneous three-particle equation derived by him is identical to the homogeneous version of present Eqs. (72) and (73), though his Born-term is different from the present work in containing Θ-functions and using $f(x_q, \vec{q}_\perp; x'_p, \vec{p}_\perp; s) = 1$. The present light-front equations (72) and (73) are also distinct from the equations of Bakker et al. [15] To the best of our knowledge the present light-front equations (72) and (73) are unique in being related to a set of instant-form three-particle equations.

IV. SUMMARY

We have provided general prescriptions for reducing four-dimensional BS like scattering integral equations to three-dimensional scattering integral equations of the LS [3] form. The resultant equations, by construction, are linear, relativistically covariant, possess correct nonrelativistic limit and yield unitary result. We have applied the present procedure to the two- and the three-particle systems. In the three-particle system the dynamical equations are written using analogy to the nonrelativistic Amado model [19] and treating the particles to be spinless bosons.

We have carried out the present procedure in two different sets of variables - the instant-form and light-front variables. Our procedure is to carry on the integration over the time component of momentum in the intermediate state phase space while assuming, as usual, that the integrand is independent of the time component of momentum. In the instant form this implies an integration over p_0, and in the light-front formalism this implies an integration over p_-. As the starting four-dimensional integral equations satisfy constraints of relativistic unitarity and covariance, the reduced equations also do so as there are no approximations which could destroy these virtues.

In the case of the two-particle system the present procedure yields the well known light-front equation first derived by Weinberg [13] and instant-form equation first derived by Blankenbecler and Sugar [5]. These two types of equations represent the same dynamics and are demonstrated to be related by a transformation of variables and hence are equivalent to each other. Consequently, the light-front two-particle equation, though not rotationally invariant in the light-front variables, will yield rotationally invariant result when expressed in instant-form variables.

The equivalence between the two types of two-particle equations allow one to perform practical partial-wave calculations using the light-front equation (26). We have shown in Sec. IID how to construct a light-front potential $V(x_q, \vec{q}; x_p, \vec{p}; s)$ from a spherically symmetric partial-wave potential $V_l(q, p, s)$ using Eq. (35). Once this potential is constructed the three-dimensional light-front equation (26) could be numerically solved to yield the same result as obtained by a direct solution of the one-dimensional partial wave instant-form equation. As the light-front equation continues to be three-dimensional, its solution may imply added numerical complications. But the final result, when expressed in terms of the physical instant-form variables, is identical to that obtained by the direct solution of the one-dimensional partial-wave instant-form equation. This is demonstrated explicitly in an analytic separable potential model. Following this procedure one can circumvent the difficulties with the angular momentum projection of the light-front dynamical equations in carrying out a numerical calculation.

In the case of the three-particle system the present procedure yields new practical three-dimensional scattering equations using the instant-form and light-front variables. We derive several instant-form equations satisfying constraints of relativistic unitarity and covariance. Of these, the sets of Eqs. (53) and (59), Eqs. (53) and (58), and Eqs. (53) and (54) are new and deserve more attention in the future. Of these sets Eqs. (53) and (54) follow as a consequence of the use of a relativistic propagator suggested long ago [9]. However, Eqs.

(53) and (54) in this form have never appeared in the literature before. The present set of light-front three-particle equations (72) and (73) are shown to be related by a transformation of variables to the instant-form equations (53) and (54), and hence the two sets represent identical dynamics. To the best of our knowledge this is the first time that a relation is established via a transformation of variables between three-particle equations in instant-form and light-front variables. The solutions of these two sets of equations are supposed to be identical as in the two-particle case. Distinct relativistic, unitary three-particle instant-form equations are provided by other authors. [6] The three-particle light-front equations (72) and (73), are, however, distinct from the three-particle light-front equations of Refs. [14,15].

The present set of instant-form three-particle equations can easily be used for studying relativistic effects on both bound state and scattering of the three-nucleon system, or studying intermediate energy pion-deuteron or pion-nucleon scattering. In the following paper we present a numerical application of the present set of equations to the study of the relativistic effect in the trinucleon system. [17]

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REFERENCES

[1] See, for example, S. K. Adhikari and K. L. Kowalski, Dynamical Collision Theory and Its Applications, Academic Press, Boston, 1991.

[2] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84 (1951) 1232.

[3] B. A. Lippmann and J. Schwinger, Phys. Rev. 79 (1950) 469.

[4] G. Rupp and J. A. Tjon, Phys. Rev. C 37 (1988) 1729; D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento 43 (1966) 258.

[5] R. Blankenbecler and R. Sugar, Phys. Rev. 142 (1966) 1051.

[6] R. Aaron, R. D. Amado, and J. E. Young, Phys. Rev. 174, (1968) 2022.

[7] G. E. Brown and A. D. Jackson, The Nucleon-Nucleon Interaction, North-Holland, Amsterdam, 1976.

[8] R. H. Thompson, Phys. Rev. D 1 (1970), 110; K. Erkelenz and K. Holinde, Nucl. Phys. A194 (1972) 161; V. G. Kadychevsky, Nucl. Phys. B6 (1968) 125; F. Gross, Phys. Rev. 186 (1969) 1448.

[9] A. Ahmadzadeh and J. A. Tjon, Phys. Rev. 147 (1966) 1111; G. Rupp and J. A. Tjon, Phys. Rev. C 45 (1992), 2133; 37 (1988) 1729.

[10] V. B. Mandelzweig and S. J. Wallace, Phys. Lett. 197B (1987) 469; S. J. Wallace and V. B. Mandelzweig, Nucl. Phys. A503 (1989) 673.

[11] B. Blankleider, Nucl. Phys. A463 (1987) 77c; T. S. H. Lee, ibid. A463 (1987), 61c; H. Garcilazo Phys. Rev. C 37(1988) 2022.

[12] H. Garcilazo, Phys. Rev. C 23 (1981) 559; H. Garcilazo, L. Mathelitsch, and H. Zankel, ibid. 32 (1985) 264.

[13] S. Weinberg, Phys. Rev. 150 (1966) 1313.

[14] M. G. Fuda, Phys. Rev. C 33 (1986) 996; 35 (1987) 226.

[15] B. L. G. Bakker, L. A. Kondratyuk, and M. V. Terentyev, Nucl. Phys. B158 (1979) 497.

[16] T. Frederico, Phys. Lett. B282 (1992) 409.

[17] S. K. Adhikari and L. Tomio, submitted to Nucl. Physics,

[18] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals Series and Products, Academic Press, New York, 1965, pages 205-206.

[19] R. D. Amado, Phys. Rev. 132 (1963) 485.

Figure Caption

1. The diagramatic form of the three-particle scattering equation (47). Each single-line represents propagation of a particle, each double-line represents propagation of the bound state or isobar of two particles.